Generators of Magnetic Groups of Symmetry and Commutation Relations

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Abstract— **A general procedure is given for determining the matrices $[S], [Z]$, and $[Y]$ of linear symmetrical multiports with gyromagnetic media. To obtain relations between the elements of the matrices, the color groups, the Curie principle, and the concept of gyrotropic symmetry (GS) and gyrotropic antisymmetry (GA) are used. Symmetries of the dc magnetic field are also considered. General properties of the multiports with GS and GA are discussed. Applications of the symmetry analysis are illustrated by two 3-D structures and some existent devices.**

I. Introduction

NONRECIPROCAL and control devices with ferrites have many applications in radioelectronics in a wide range of frequencies, both in radio and optical domains. Among these devices, there are reciprocal and nonreciprocal microwave two-, three-, four-, and multiports, such as isolators, phase shifters, circulators, switchers, nonreciprocal dividers/combiners, control directional couplers, etc.. Some of them fulfill several functions simultaneously; e.g., dividing/combining and isolating [1], [2], dividing and adjustable phase shifting [3], and isolating and transforming of impedances [1], [2]. Several nonreciprocal elements with magnetooptic materials have been proposed in optics. Many of these devices with good performances are available commercially. A vast number of papers and many books are devoted to the subject.

These devices are based on different physical effects and on the use of various transmission lines. Some of them have 3–D structures. Rigorous electrodynamic methods of analyzing such devices are often very complicated. At the same time, considerable results can be obtained by using the network theory. Matrix methods with the condition of unitarity allow us to find out some general properties of multiports without having to solve Maxwell's equations, to define realizability of different functions, etc. Unfortunately, in many cases a general matrix analysis of multiports is difficult because of the large number of parameters of their matrices.

Considerable simplification of the problem is achieved by the use of the theory of symmetry. Most existent devices exhibit different types of spatial (geometrical) symmetry-mirror reflection in a plane, rotation and others. By virtue of symmetry analysis, the number of independent elements of the matrices (scattering $[S]$, impedance $[Z]$ and admittance $[Y]$) may be reduced significantly. Intensive work in the analysis of symmetrical reciprocal waveguide junctions has been made by Montgomery *et al.* [4] and thoroughly carried out by Kerns [5].

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When ferrites appeared in the microwave technology, the theory of symmetry and the apparatus of point groups have been used to analyze and synthesize the N-port circulators [6]. This approach does not differ from that being used for the reciprocal devices. Bidirectionality and mode orthogonality in nonreciprocal waveguides have been treated by Mc Isaac [7], [8]. The $[S]$-matrix for the two-port with gyrotropic media has been written in [9].

In the 1960's and 1970's, works concerning symmetry of the electromagnetic field in gyrotropic media were accomplished by Kong [10], Someda [11], Rao and Wu [12], and others. Later, Altman, Schatzberg, and Suchy [13], [14] investigated symmetry transformations and time reversal of currents and fields in (bi)anisotropic media in a generalized form.

The problem of symmetry properties of the devices with gyromagnetic media is more complicated than that of the devices with isotropic media. The device with gyromagnetic media presents a geometric-physical object which has a certain geometrical symmetry and a symmetry of the magnetization (or external biasing magnetic field $\bar{H}_0$). Hence, we have to consider the symmetry of the system: the geometrical structure plus magnetic field.

One type of the possible symmetries has been considered in the classical paper [6], namely, the N-fold rotational one. It is a particular case of the general one called gyrotropic symmetry (GS) [15], in which $\bar{H}_0$ and the electromagnetic field are invariant under a group of symmetry operations. So, using the usual commutation relations, we may find some relations between elements of the matrices $[S], [Z], [Y]$.

It has been shown in [15] and [16] that in another case, which has been called gyrotropic antisymmetry (GA), under fulfillment of certain requirements we can also find some relation between the elements of the matrices, although $\bar{H}_0$ and the electromagnetic fields are not invariant under geometrical symmetry operations. In this case, we need to consider the adjoint Maxwell's equations in the adjoint media. In some devices, GS and GA are met simultaneously and the use of both of them gives additional information.

II. Problem Formulation and Contents of the Paper

We shall treat N-ports with a geometrical symmetry, biasing by a direct magnetic field, which also has a certain symmetry. A media which fills the N-port, is in general a nonhomogeneous, gyromagnetic one with tensor properties. Symmetry of the parameters of the media is consistent in a certain way with the geometrical symmetry of the device. The media may be with or without losses. Radiation from the N-port is permitted. The restriction which is imposed on the media is its linearity. Several modes can propagate in every physical port in accordance with the network theory. In this case, we are to consider a corresponding mode base of the device. We shall use the classical network theory [9].

The purpose of the present work is a theory of such N-ports, which is developed with the use of the color group approach. The next section is devoted to a brief discussion of color group properties. The conditions of GS and GA will be presented in Section IV. Symmetries of the dc magnetic field, which were not given attention earlier, are discussed in Section V. Section VI shows some properties of N-ports with GS and GA. Applications of the theory to different structures, including existent devices, are given in Section VII.

III. Magnetic Point Groups of Symmetry

Symmetry operations bring the device into self-coincidence. If the device contains only a media with symmetrical tensor parameters, the geometrical symmetry of the device leads to physical symmetry, i.e., to the symmetry of the e.m. fields in the symmetrical points. As a result, we may find certain relations between the $[S]$-elements [4].

If the device is filled with a gyromagnetic media (the permeability is nonsymmetrical tensor), in addition to a geometrical symmetry of the device, it is necessary to take into account the symmetry of the magnetic parameters of the media, which in turn depends on the symmetry of the external magnetic field $\bar{H}_0$. In general, if the structure of $\bar{H}_0$ is arbitrary, the theory of symmetry cannot be applied. But luckily, most devices are magnetized by $\bar{H}_0$, which is consistent with the geometrical symmetry of the device. It is often required to provide the necessary conditions for the physical phenomenon that is used in the device.

There are three cases in which the theory of symmetry can help to simplify the $[S]$-matrix. In the first GS-case, the symmetry of $\bar{H}_0$ corresponds to the geometrical symmetry of the device. (Notice that the symmetry of $\bar{H}_0$ is defined by taking into account its axial nature). In this case, symmetry of the e.m. field exists, and we can find certain relations between some elements of $[S]$ using the method of the isotropic variant.

In the second GA-case, there is no usual symmetry of $\bar{H}_0$ and no symmetry of the e.m. field at all. Nevertheless, we may get useful information about the $[S]$-matrix. The conditions under which the two cases exist will be described in the next section.

In the third case, both GS and GA are presented in one device. Notice that in general the device is nonreciprocal in every one of the three cases.

It is convenient to discuss symmetry properties of a device using group-theoretical technique, especially when the device has a high level of symmetry. There are many books on group theory [18]–[20] and some papers concerning microwave junctions based on the theory [5], [11]. Therefore, it will not be given here. We shall consider only some relevant information about magnetic point groups.

We often need to consider the reversal of $\bar{H}_0$ (or the magnetization M produced by the magnetic field) in the N-ports with gyromagnetic media, in addition to the usual geometrical operations such as rotation, reflection, and inversion. The reversal of $\bar{H}_0$ is caused by the reversal of the current direction under the time reversal operation T.

There are three categories of magnetic point groups, which are a particular case of Shubnikov, or color groups [21]:

1) the groups containing operation T as an element;
2) the groups that do not include T at all; and
3) the groups containing T in combination with geometrical operations.

We shall primarily be interested in the third category of the groups and partly in the second one, because the first category of the groups describes the devices without gyrotropic materials. The devices described by the second category are investigated by standard methods. Such devices are, for example, classical three- and four-port circulators with axial symmetry.

Let us enumerate some properties of magnetic groups of the third category [20]. Let A_i, A_k be geometrical operations of symmetry. A point magnetic group may contain two types of elements, with and without T. The former $M = TA_k$, sometimes referred to as antioperators, and the latter A_i do not contain time reversal. T itself is not a member of the point group but must occur only in combination with other operators. This restriction eliminates the elements of odd-order in rotation operations from magnetic groups; for instance, TC_3 (here C_3 is three-fold rotation symmetry in the Schoenflies notation). Notice that there cannot exist groups with elements with T only, because a unit element E, which is in every group, cannot be with T.

The sets of A_i and A_k are different; that is, no elements A_j appear both with and without T. If the whole magnetic group is $G = \{H, M_k\}$, where $H = \{A_i\}, M_k = \{TA_k\}$ (here the symbol $\{\ \}$ denotes a rotation-reflection group), H forms a subgroup of G, and the number of elements A_i must be equal to the one of TA_k. Replacing T by E, one obtains the set $G' = \{A_i, A_k\}$, which is also a group. In Schoenflies notation, the magnetic group is denoted by $G'(H)$.

In any group the law of multiplication is fulfilled, so that the product A_iA_k of any two elements is also an element of the group. We see that the product of A_i and TA_k is an element with T, but the product of TA_j and TA_k gives an element without T

$$TA_jTA_k = A_jT^2A_k = A_jA_k$$

because T commutes with any ordinary operation and $T^2 = E$.

The operations without T correspond to GS and the operations with T to GA. The groups which describe the devices with GS only do not contain elements with T, and they belong to the second category of magnetic groups.

One of the important questions in our discussion is generators of the symmetry groups. To find all the elements of a group, it is possible to use a small number of elements known as generators. A definition of generators is given in [19]: "A set P of elements of a group G is a system of generators of the group if every element of G can be written as the product of a finite number of factors, each of which is either an element of P or the inverse of such an element." Detailed description of generators with their tables for many point groups is given in [19].

Further, every generator may be presented by its equivalent $N \times N$ matrix representation $[R]$, or the symmetry operator of the device. The structure of $[R]$ and its dimension depend on the mode bases of the device, that is, on the number of ports and the number of modes, taking into consideration modes in every port (the dimensions of $[S]$- and $[R]$-matrices are equal). Each row and each column of the matrix $[R]$ contains only one nonzero element, $+1$ or -1. These representations, or the symmetry operator matrices will also be further called generators.

IV. Conditions for GS and GA

Let $[\gamma]$ be a mapping operator. In the 3–D space with the position vector $\bar{\mathbf{r}}(x, y, z)$, it may represent any rotation, reflection, and inversion. This operator connects the given points $\bar{\mathbf{r}}(x, y, z)$ and the mapped (symmetrical) ones $\bar{\mathbf{r}}'(x, y, z)$

$$\bar{\mathbf{r}}' = [\gamma]\bar{\mathbf{r}}.$$

For example, for reflection in the plane $x = 0$, the operator $[\gamma]$ takes the form

$$[\gamma] = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Consider two cases that are met under symmetrical transformations of Maxwell's equations [13], [14].

1) In the mapped points $\bar{\mathbf{r}}'$, Maxwell's equations remain in the same form, i.e., the equations are invariant under these transformations. The relation between the medias parameters in the points $\bar{\mathbf{r}}'$ and $\bar{\mathbf{r}}$ that must be fulfilled is

$$[\mu(\bar{\mathbf{r}}')] = [\gamma][\mu(\bar{\mathbf{r}})][\gamma] \tag{1}$$

where $[\mu]$ is the permeability tensor. This is the case of GS. The commutation relation for the scattering matrix $[S]$ and a generator $[R]$ is

$$[R][S] = [S][R]. \tag{2}$$

2) In the mapped points, Maxwell's equations become the adjoint ones. It is possible under the condition

$$[\mu(\bar{\mathbf{r}}')] = [\gamma][\mu(\bar{\mathbf{r}})]^t[\gamma] \tag{3}$$

where t denotes transposition. This is the case of GA and the identity

$$[R][S] = [S]^t[R] \tag{4}$$

is valid. The proof of (4) is given in [16].

It is shown in [23] that the dissipative parameters of a gyromagnetic media and its adjoint are the same, so we may treat a lossy media.

V. Symmetries of DC Magnetic Field

Consider some examples of the fields that are met in real devices.

A. Uniform Field with Constant Direction [Fig. 1(a)]

It is met in many devices, for instance in the classical three- and four-port circulators. This approximation is used when the dimensions of the device are less than those of the pole-shoes of the magnet. Using Schoenflies notation, we may describe the field as follows. It has ∞-fold principal rotation axis, which is along the z-axis with C_∞ symmetry, σ_Z-reflection in a plane $z = 0$ perpendicular to the principal axis. Inversion i may be defined as $i = C_2\sigma_Z$. Besides, in the magnetic group there are the infinite set of antiplanes of symmetry $T\sigma_V$, which pass through the principal axis, and the infinite set of two-fold antiaxes, which lie in the plane $z = 0$ with symmetry TC_2.

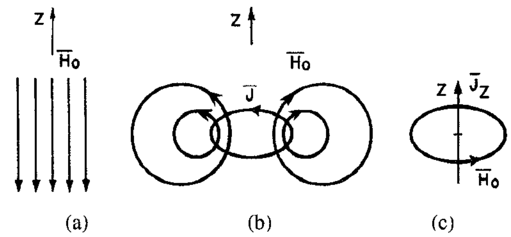


Fig. 1. Structures of dc magnetic fields in ferrite devices.

Hence, with identity operation E, we may write down the elements of the group

$$E, C_\infty, \sigma_Z, i, TC_2, T\sigma_V. \tag{5}$$

B. The Field $\bar{H}_0$ Is Produced by a Current Loop $\bar{J}$[Fig. (1b)]

This case is met in the classical ferrite switchable junction circulators [26]. The symmetry of this field coincides with the one of case 1.

C. The Ring Field Is Produced by a Current Line $\bar{J}_z$ [Fig. 1(c)]

It is met, for example, in the ferrite phase shifters. Let us enumerate the elements of the symmetry in Schoenflies notation (z is the principal axis)

$$E, C_\infty, \sigma_V, TC_2, T\sigma_Z, Ti. \tag{6}$$

Comparing (5) and (6) shows that the elements that coincide in cases 1 and 3 are E, C_∞, and TC_2. The other three elements differ by the time reversal operation.

Dc magnetic fields with other symmetries are used in devices as well. In [25] for example, the field $\bar{H}_0$ has antiparallel directions in two halves of the device; in [26] a device with a quadrupole field is considered, in [17]-field with linear dependence with respect to one coordinate.

VI. Properties of Matrices of Multiports with GS and GA

Consider first ferrite N-ports with a symmetry plane. In this case, the matrix $[R]$ is symmetrical.

A. The Case of GS

In the most simple variant, the dc magnetic field is perpendicular to the plane of symmetry. From the identity (2) for the multiport on Fig. 2 with the plane of symmetry $y = 0$, we may write down relations for some pairs of ports

$$S_{2,3} = S_{N,N-1}, \quad S_{3,2} = S_{N-1,N},$$
$$S_{2,N-1} = S_{N,3}, \quad S_{N-1,2} = S_{3,N},$$
$$S_{1,2} = S_{1,N}, \quad S_{2,1} = S_{N,1}, \quad \text{and so on.}$$

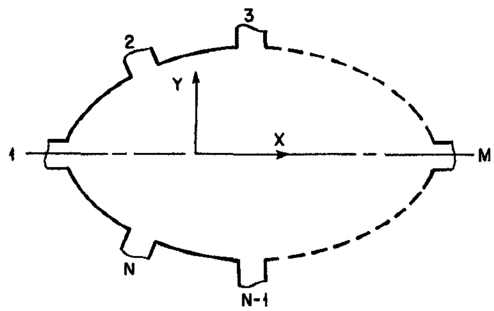


Fig. 2. N-port with a plane of symmetry.

The relations between the elements $S_{1,2}$ and $S_{2,1}, S_{2,3}$ and $S_{3,2}$ and so on, are not determined. Hence, in general, these connections are nonreciprocal.

Consider S_{ll} and S_{mm} in two symmetrical ports l and m, for example, $l = 2, m = N$. The elements of the matrix $[R] R_{lm} = R_{ml} = \pm 1$. From the symmetry, the signs of R_{lm} and R_{ml} must be the same, and

$$([R][S])_{lm} = \sum_{j=1}^{N} R_{lj} S_{jm} = \pm S_{mm}$$

since all the elements $R_{lj} (j = 1, 2, 3 \dots N)$ in the row 1 are equal to 0 except R_{lm}

$$([S][R])_{lm} = \sum_{j=1}^{N} S_{lj} R_{jm} = \pm S_{ll}$$

since all the elements $R_{jm} (j = 1, 2, 3 \dots N)$ in the column m are equal to 0 except R_{lm}. So from (2) we get $S_{ll} = S_{mm}$. The same result is obtained from $([R][S])_{ml}$ and $([S][R])_{ml}$. Hence the scattering coefficients S_{ll} and S_{mm} of symmetrical ports are equal.

Consider now relations between the elements S_{lm} and S_{ml}, where l and m correspond symmetrical ports. Define the diagonal elements of the matrix of the product $[S]$ and $[R]$

$$([S][R])_{ll} = \sum_{j=1}^{N} S_{lj} R_{jl} = \pm S_{lm}$$

$$([R][S])_{ll} = \sum_{j=1}^{N} R_{lj} S_{jl} = \pm S_{ml}.$$

From (2) we can make a conclusion that $S_{lm} = S_{ml}$, so that the connection between symmetrical ports is reciprocal. The same result can be gotten from the consideration of the elements $([S][R])_{mm}$ and $([R][S])_{mm}$.

At first sight, the equality $S_{lm} = S_{ml}$ looks rather strange because with $\bar{H}_0$ being perpendicular to the plane of symmetry, Faraday's effect is possible, which is nonreciprocal. An explanation of this phenomena is as follows. First, the e.m. fields in l and m ports are symmetrical [11] because the sense of Faradays rotation does not depend on the sense of wave propagation. We use an approximation of the network theory with the definition of $[S]$ in terms of equivalent voltage waves [9]. So under this description, the nonreciprocal polarization properties of the e.m. waves are lost. To take into consideration the polarization effects, we should consider instead of the two ports l and m, four ports with two orthogonal polarizations (an example is shown in Section VII). Second, to obtain a nonreciprocal device with Faraday's effect, we have to bring an asymmetry into the device (a twisted waveguide section or two grids with nonsymmetrically oriented grating patterns). However, in this case, we cannot use the mirror symmetry.

Examine now two ports l and m lying in the plane of symmetry . They are l and M in Fig. 2. The corresponding elements are on the main diagonal of $[R]$ and the signs of R_{ll} and R_{mm} may be the same or different. The elements S_{lm} and S_{ml} in the product of $[S]$ and $[R]$ are in the places lm and ml, respectively, and

$$([S][R])_{lm} = \sum_{j=1}^{N} S_{lj} R_{jm} = \pm S_{lm}, \quad ([R][S])_{lm} = \pm S_{lm},$$
$$([S][R])_{ml} = \pm S_{ml}, \quad ([R][S])_{ml} = \pm S_{ml}.$$

So if the signs of R_{ll} and R_{mm} are the same, the connection between the ports l and m is not determined. But if the signs of R_{ll} and R_{mm} are different, from (2) we get $S_{lm} = -S_{lm}, S_{ml} = -S_{ml}$, and a connection between these ports is absent ($S_{lm} = S_{ml} = 0$). It is easy to show that a connection between the scattering coefficients S_{ll} and S_{mm} is not determined.

B. The Case of GA

In this case, $\bar{H}_0$ can be, for instance, parallel to the plane of symmetry $y = 0$ (Fig. 2). In general, the e.m. fields in symmetrical ports are not symmetrical. From (4) we may write some relations for pairs of ports

$$S_{2,3} = S_{N-1,N}, \quad S_{3,2} = S_{N,N-1},$$
$$S_{2,N-1} = S_{3,N}, \quad S_{N-1,2} = S_{N,3},$$
$$S_{1,2} = S_{N,1}, \quad S_{2,1} = S_{1,N}, \quad \text{and so on.}$$

Bearing the symmetry of $[R]$ in mind, the right-hand side of (4) may be transformed as follows

$$[S]^t [R] = [S]^t [R]^t = ([R][S])^t$$

and (4) takes the form

$$[R][S] = ([R][S])^t. \tag{7}$$

Hence, the product of the matrices $([R][S])$ must be symmetrical, and the number of independent parameters of $[S]$ is reduced to $N(N+1)/2$.

If two ports l and m are symmetrical in a geometrical sense, the elements S_{lm} and S_{ml} in $([R][S])$ are on the main diagonal and they remain in the same places under transposition. Hence from (4), it is not possible to find a type of connection between l and m ports. But for the reflection coefficients, we may write $S_{ll} = S_{mm}$.

Consider now two ports l and m lying in the antiplane of symmetry. Scrutiny of (7) shows that we fail to find a relation between S_{ll} and S_{mm}. Considering S_{lm} and S_{ml}, we have

$$([R][S])_{lm} = \sum_{j=1}^{N} R_{lj} S_{jm} = \pm S_{lm}, \quad ([R][S])_{ml} = \pm S_{ml}.$$

In this case, if R_{ll} and R_{mm} have the same signs, the connection between the ports l and m is reciprocal and

TABLE I
SYMMETRY OF THE ORTHORHOMBIC STRUCTURE WITH UNIFORM MAGNETIC FIELD

	Elements of symmetry							
Direct magnetic field $\bar{H}_0$	E	C_∞	$T\sigma_v$		TC_2		σ_z	i
Geometrical structure	E	C_{2z}	σ_x	σ_y	C_{2x}	C_{2y}	σ_z	i
Resultant symmetry	E	C_{2z}	$T\sigma_x$	$T\sigma_y$	TC_{2x}	TC_{2y}	σ_z	i

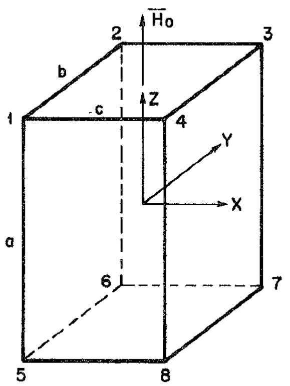


Fig. 3. Orthorhombic structures with uniform dc magnetic field along z-axis.

$S_{lm} = S_{ml}$. If R_{ll} and R_{mm} have different signs, the equality $S_{lm} = -S_{ml}$ is valid. This relation may be explained as follows. If there is a component of $\bar{H}_0$ parallel to the line connecting the ports l and m, Faraday's effect is possible. It leads to nonreciprocal π-phase shift between these ports, one of them is connected parallel and the other-series to the junction.

We may get similar results for $[S]$-elements in the cases with axial and inversion symmetries. It should be reminded that GA is possible with an axis of even order, and the matrix $[R]$ is symmetrical in the case of rotating through π. For inversion symmetry, $[R]$ is always symmetrical.

VII. APPLICATIONS

A. Application to Orthorhombic Structure

To demonstrate the basis of the method, let us consider a hypothetical orthorhombic structure (Fig. 3) with a b c. The structure is geometrically invariant under rotation, reflection, and inversion. The ports of the structure which are connected with the verteces 1–8, may be implemented for instance as coaxial or circular waveguides with only one propagating mode in every port.

If no external magnetic field is applied, the geometrical symmetry of the structure is D_{2h}. The group of symmetry consists of 8 elements [20]: E-identity; C_{2x}, C_{2y}, C_{2z}-rotations around x, y, z axes, respectively; $\sigma_x, \sigma_y, \sigma_z$-reflections in the planes $x = 0, y = 0$, and $z = 0$, respectively; and i-inversion.

Generators of the group D_{2h} may be, for instance, C_{2x}, C_{2z}, and σ_z. Using these generators, we can get the rest of the elements: $E = \sigma_z^2, C_{2y} = C_{2x}C_{2z}, i = C_{2z}\sigma_z, \sigma_y = C_{2x}\sigma_z, \sigma_x = C_{2x}i$. Note that we may choose another set of independent generators, e.g., $C_{2z}, \sigma_x, \sigma_z$.

The knowledge of generators allows us to find 8×8 generating matrices $[R]_{C_{2X}}, [R]_{2Z}, [R]_{\sigma Z}$ of the structure, which are representations of the corresponding symmetry operators. Write, for instance, the matrix $[\mathbf{R}]_{C_{2X}}$

$$[\mathbf{R}]_{C_{2X}} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Apply to a case with dc magnetic field. Let $\bar{\mathbf{H}}_0$ be along the z-axis (Fig. 3). Comparing the symmetry group D_{2h} and the magnetic symmetry group of $\bar{H}_0$ (Section V) and using the Curie principle of symmetry superposition (only those elements remain in the resultant symmetry which are common for the whole system [21]) we can write down the magnetic symmetry group of the system: geometrical structure plus magnetic field $\bar{H}_0$. This group $D_{2h}(C_{2h})$ contains the following elements

$$E, C_{2z}, \sigma_z, i, TC_{2x}, TC_{2y}, T\sigma_x \text{ and } T\sigma_y.$$

The elements $T\sigma_v$ and TC_2 of the group of $\bar{H}_0$ are disintegrated into two elements each (Table I).

Using the set of generators $[\mathbf{R}]_{C_{2X}}, [\mathbf{R}]_{C_{2Y}}$, and $[\mathbf{R}]_{\sigma_Z}$, we must take into account that x is an antiaxis of symmetry. Hence, to find the matrix $[S]$, we should use the identities

$$[\mathbf{R}]_{C_{2Z}}[\mathbf{S}] = [\mathbf{S}][\mathbf{R}]_{C_{2Z}} \tag{8}$$

$$[\mathbf{R}]_{\sigma Z}[\mathbf{S}] = [\mathbf{S}][\mathbf{R}]_{\sigma Z} \tag{9}$$

$$[\mathbf{R}]_{C_{2X}}[\mathbf{S}] = [\mathbf{S}]^t[\mathbf{R}]_{C_{2X}}. \tag{10}$$

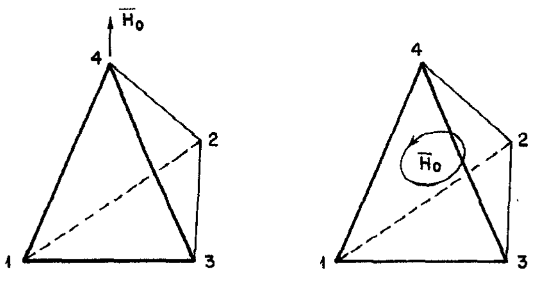


Fig. 4. Regular triangular pyramid. (a) With uniform dc magnetic field. (b) With ring magnetic field.

Two former identities coincide with those for the nongyromagnetic structure, but the latter has another form. Equations (8) and (9) correspond to the case of GS and (10) to one of GA.

The $[S]$-matrix defined from (8)–(10) has 12 independent parameters

$$[\mathbf{S}] = \begin{vmatrix} [\mathbf{S}]_1 & [\mathbf{S}]_2 \\ [\mathbf{S}]_2 & [\mathbf{S}]_1 \end{vmatrix}$$

$$\times \begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} \\ S_{21} & S_{11} & S_{23} & S_{13} & S_{25} & S_{15} & S_{27} & S_{17} \\ S_{13} & S_{14} & S_{11} & S_{12} & S_{17} & S_{18} & S_{15} & S_{16} \\ S_{23} & S_{13} & S_{21} & S_{11} & S_{27} & S_{17} & S_{25} & S_{15} \\ S_{15} & S_{16} & S_{17} & S_{18} & S_{11} & S_{12} & S_{13} & S_{14} \\ S_{25} & S_{15} & S_{27} & S_{17} & S_{21} & S_{11} & S_{23} & S_{13} \\ S_{17} & S_{18} & S_{15} & S_{16} & S_{13} & S_{14} & S_{11} & S_{12} \\ S_{27} & S_{17} & S_{25} & S_{15} & S_{23} & S_{13} & S_{21} & S_{11} \end{vmatrix}.$$

B. Application to Regular Triangular Pyramid

Apply now to a more realistic example of a four-port volume structure (Fig. 4), which may be used as a nonreciprocal three-way power divider. One example of the possible realizations is given in [22]. It has the main (input) coaxial port four and three output microstrip ports one, two, and three with the edge-guided modes.

Let the external magnetic field be applied along the axis of symmetry, which goes through vertex 4 [Fig. 4(a)] and the system has only the three-fold axis of symmetry. Generator of the group C_3 is

$$[R]_{C_3} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{11}$$

It is a case of GS, so we use (2) and find $[S]$ with six independent parameters

$$[S] = \begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{13} & S_{11} & S_{12} & S_{14} \\ S_{12} & S_{13} & S_{11} & S_{14} \\ S_{41} & S_{41} & S_{41} & S_{44} \end{vmatrix}. \tag{12}$$

Analysis of (12) shows that because of $S_{14} \neq S_{41}$, the structure may on principle be used as a nonreciprocal one with isolating and/or phase-shifting properties. The 3×3 matrix in the upper left corner of $[S]$ corresponds to the matrix of a three-port circulator.

Now, we add to the system three antiplanes of symmetry (GA) passing through the three-fold axis. Generators may be chosen $[R]_C$ (11) and $[R]_\sigma$

$$[R]_\sigma = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

The matrix $[S]$ now has five independent parameters

$$[S] = \begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{13} & S_{11} & S_{12} & S_{14} \\ S_{12} & S_{13} & S_{11} & S_{14} \\ S_{14} & S_{14} & S_{14} & S_{44} \end{vmatrix}$$

and we cannot get nonreciprocal connections between the ports one, two, three, and the port four. It is in accordance with the conclusions of Section VI because, for example, ports one and four lie in an antiplane of symmetry.

Consider now the pyramid which is biased with a ring magnetic field [Fig. 4(b)]. If the structure has only the three-fold axis of symmetry (GS), we get the matrix (12). If the structure in addition to the axis possesses three planes of symmetry (GS), the matrix $[S]$ turns out to be

$$[S] = \begin{vmatrix} S_{11} & S_{12} & S_{12} & S_{14} \\ S_{12} & S_{11} & S_{12} & S_{14} \\ S_{12} & S_{12} & S_{11} & S_{14} \\ S_{41} & S_{41} & S_{41} & S_{44} \end{vmatrix} \tag{13}$$

with 5 parameters. We see that in this case, the upper left 3×3 matrix represents the matrix of a symmetrical reciprocal three-port (so if we consider a symmetrical three-port with three planes of symmetry and with a ring magnetic field, it exhibits only reciprocal properties; it is also true for any regular plane polygon).

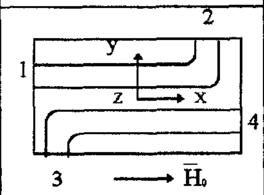
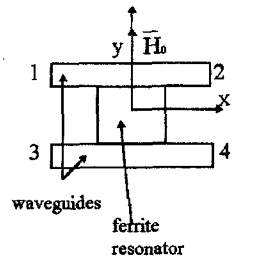
C. Two Ports

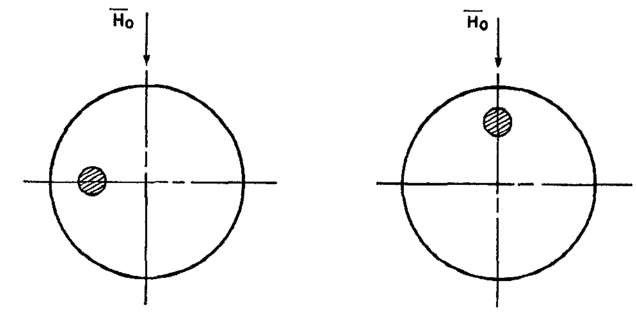
Two examples of two-ports with planes of GS and GA will be given in this subsection.

1) Circular Waveguides with Ferrite Rods: There are planes of symmetry in both waveguides on Fig. 5(a) and (b). In the left one, it is GS, and in the right waveguide-GA. In both cases, ferrite rods are in the planes. In accordance with the results of Section VI, for GS a nonreciprocal effect is possible. But for GA, it is not. It has been mentioned in [27] that in the waveguide in Fig. 5(a), the phase shift is nonreciprocal. But in the waveguide in Fig. 5(b), the phase shift is reciprocal.

2) Waveguides with Two Ferrite Elements: Two two-ports with ferrites are shown in Fig. 6. The first one (Fig. 6(a)) is the usual rectangular waveguide with two ferrite elements, 1 and 2. The plane of geometrical symmetry $x = 0$ corresponds to GA so that the "directions" of nonreciprocal effect in two ferrite elements situated in the left and right halves of the waveguide are opposite. The second two-port (Fig. 6(b)) is the Karp slow-wave ladder structure with isolating performance [28]. The dc magnetic field here is perpendicular to the plane of symmetry $x = 0$, so that it is a case of GS and the "directions" of the nonreciprocal effect in the ferrite elements

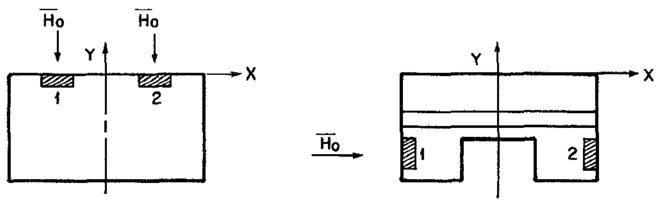
TABLE II
CHARACTERISTICS OF EXISTENT DEVICES

Type of device	Schematic diagram	Elements of symmetry	Group of symmetry	Generators	[S]-matrix	Number of independent parameters of [S]
a) coupled slot-line 4-port circulator[30]		E, TC_2	$C_2(C_1)$	$[R]_{C_2} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{13} \\ S_{31} & S_{32} & S_{22} & S_{12} \\ S_{41} & S_{31} & S_{21} & S_{11} \end{vmatrix}$	10
b) nonreciprocal tunablele directional filter [31]		E, σ_Y, $T\sigma_X$, TC_2	$C_{2V}(C_S)$	$[R]_{\sigma_X} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$ $[R]_{\sigma_Y} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{11} & S_{23} & S_{13} \\ S_{13} & S_{14} & S_{11} & S_{12} \\ S_{23} & S_{13} & S_{21} & S_{11} \end{vmatrix}$	6



(a) (b)

Fig. 5. Circular waveguides with a ferrite rod.



(a) (b)

Fig. 6. Two-ports with two ferrite elements. (a) Rectangular waveguide. (b) Slow-wave ladder structure.

1 and 2 are the same. So the concept of GS and GA may also help in those cases when we need to define the "direction" of nonreciprocal effect in symmetrical devices with several ferrite elements.

D. Application to Some Existent Devices

In Table II, a compressed description of several ferrite devices with different symmetries is presented. For every device, its elements of symmetry, generators, $[S]$-matrix, and the number of independent parameters have been derived.

1) The coupled slot-line four-port circulator has only one two-fold antiaxis of symmetry (GA).
2) This four-port has one plane (GS), one antiplane (GA) and one antiaxis of symmetry (GA).

VIII. DISCUSSION

The previous discussions have been concerned with the matrix $[S]$. It is not difficult to show that all the results of the paper are valid for $[Z]$ and $[Y]$ matrices as well. Substitute, for example, the relation

$$[S] = ([Z] - [E])([Z] + [E])^{-1}$$

into the identity (4) and make some algebraic transformations

$$\begin{aligned} &[R]([Z] - [E])([Z] + [E])^{-1} \\ &\quad = (([Z] - [E])([Z] + [E])^{-1})^t [R], \\ &[R]([Z] + [E] - 2[E])([Z] + [E])^{-1} \\ &\quad = ([Z]^t + [E])^{-1}([Z]^t + [E] - 2[E])[R], \\ &[R]([Z] + [E])^{-1} = ([Z]^t + [E])^{-1}[R]. \end{aligned}$$

Multiplying the last equation from the left by $([Z]^t + [E])$ and from the right by $([Z] + [E])$, we get

$$[R][Z] = [Z]^t[R].$$

The identity

$$[R][Y] = [Y]^t[R]$$

can be proved in a similar way.

Notice that both GS and GA cases are reduced, of course, to a case with isotropic media ,and the correspondent matrices are symmetrical when $\bar{H}_0$ becomes null, but GS and GA are not reduced to each other.

It is of interest to note that in the cases without a plane of symmetry, there are so-called enantiomorphous modifications

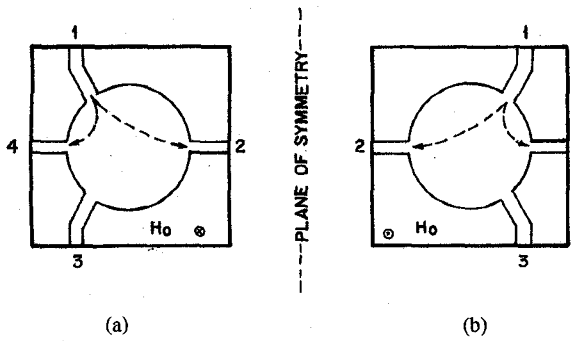


Fig. 7. The enantiomorphous modifications of a two-way divider/combiner.

[21] of the devices. Consider, for instance, a microstrip divider/combiner with the disc resonator [1], shown in Fig. 7(a). It works under certain direction of $\bar{H}_0$. To get the device with the opposite direction of $\bar{H}_0$, we must take the enantiomorphous modification of the device, i.e., its mirror reflection (Fig. 7(b)).

The following remark should be made as well. The symmetry approach used in the paper, has an abstract nature. It gives only necessary conditions for realization of nonreciprocal effects in devices. Sufficient conditions may be received only as a result of Maxwell's equations solving. But if the theory shows impossibility of nonreciprocal effects (i.e., it shows the reciprocity), it is a sufficient condition.

IX. Conclusion

The main aim of this paper is to show a method of finding the matrices $[S]$, $[Z]$, and $[Y]$ for symmetrical devices with gyromagnetic media. It allows us to reduce the number of independent parameters of the matrices. The approach given in this paper is applicable to devices with different symmetries and medias that comply with (1) or (3), with the restriction of the linearity of the medias. The method is applicable also for the devices with gyroelectric media. It may be helpful especially for the devices with complicated geometrical structures and magnetic fields.

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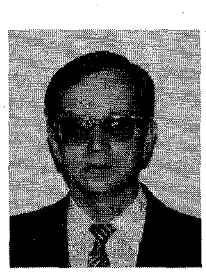

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